On the Additive Structure of Cyclic Semirings: The Size of Their Sets of Atoms and Strong Atoms

(Joint work with Jiya Dani, Marly Gotti, Bryan Li, Arav Paladiya, Joseph Vulakh)

Anna Deng and Jason Zeng

(Mentors: Felix Gotti & Marly Gotti)

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Outline

Some Notation and Reminders

Some Notation Adopted Here

- $\mathbb{N} := \{1, 2, \ldots\}$.
- $\mathbb{N}_0 := \{0\} \cup \mathbb{N} = \{0, 1, 2, \ldots\}$.
- $\mathbb{P}$ denotes the set of standard primes.
- $\mathbb{A}$ denotes the set of algebraic complex numbers.

Commutative Monoids

Definition: A set M equipped with a binary operation $* : M \times M \to M$ is called a monoid if the following two conditions hold:

- $*$ is associative, that is, $(a * b) * c = a * (b * c)$ for all $a, b, c \in M$, and
- there exists $e \in M$ such that $e * a = a * e = a$ for all $a \in M$, in which case e is unique and called the identity element of M.

Assume now that M is a monoid under the binary operation $*$.

- M is commutative if $a * b = b * a$ for all $a, b \in M$.
- M is cancellative if for all $a, b, c \in M$, the equality $a * b = a * c$ implies that $b = c$.
- When $*$ is denoted by $+$ (resp., $\cdot$), we call M an additive monoid (resp., a multiplicative monoid), in which case, e is denoted by 0 (resp., 1).

Conventions

IMPORTANT CONVENTIONS WE KEEP THROUGH THE PRESENTATION

Two Conventions

- All monoids we deal with in this presentation are assumed to be commutative.
- Unless we specify otherwise, monoids here are assumed to be **additive and cancellative**.

Examples:

1. $\mathbb{N}_0, \mathbb{Q}_{\geq 0}$ and, $\mathbb{R}_{\geq 0}$ are additive monoids under the standard addition.
2. $\mathbb{N}, \mathbb{Q}_{\geq 1}$, and $\mathbb{R}_{\geq 1}$ are multiplicative monoids under the standard multiplication.

Submonoids and Generating Sets

Definition: Let M be an additive monoid, and let S and N be subsets of M.

- N is called a submonoid of M if $0 \in N$ and N is closed under addition.
- $\langle S \rangle$ denotes the intersection of all the submonoids of M containing S and is called the submonoid of M generated by S.
- $\langle S \rangle = \left\{ s_1 + s_2 + \cdots + s_n : n \in \mathbb{N}_0 \text{ and } s_1, \ldots, s_n \in S \right\}$.
- $\langle S \rangle$ is the smallest submonoid of M containing the set S.
- S is called a generating set of M if $M = \langle S \rangle$.
- M is finitely generated if $M = \langle S \rangle$ for some finite subset S.

Examples:

1. The monoid $\{0\} \cup \mathbb{N}_{\geq 2}$ is generated by the set $\{2, 3\}$.
2. In the additive monoid $\mathbb{Q}_{\geq 0}$, the submonoid generated by the subset $\left\{ \frac{1}{2^k} : k \in \mathbb{N}_0 \right\}$ is $M := \left\{ \frac{n}{2^k} : n, k \in \mathbb{N}_0 \right\}$.

   Note: M is not finitely generated because $(\inf M \setminus \{0\}) = 0$.

More Examples of Monoids: Numerical and Puiseux Monoids

Definition: A submonoid N of the additive monoid $\mathbb{N}_0$ (under the standard addition) is called a numerical monoid if the set $\mathbb{N}_0 \setminus N$ is finite.

Remark: A submonoid of $\mathbb{N}_0$ is a numerical monoid if and only if 1 is the only positive integer that is a common divisor of N.

Examples:

1. For all $n \in \mathbb{N}$, the monoid $\{0\} \cup \mathbb{N}_{\geq n}$ is a numerical monoid, generated by the set $\{n, n+1, \ldots, 2n-1\}$.
2. Given $a, b, k \in \mathbb{N}$ with coprime a and b, we can check that the only positive common divisor of the set $\{a^i b^{k-i} : i \in \{0, 1, \ldots, k\}\}$ is 1, whence $\langle a^i b^{k-i} : i \in \{0, 1, \ldots, k\}\rangle$ is a numerical monoid.

Definition: Any submonoid of the additive monoid $\mathbb{Q}_{\geq 0}$ is called a Puiseux monoid.

Examples:

1. $\{0\} \cup \mathbb{Q}_{\geq 1}$ is a Puiseux monoid.
2. $\langle q^n : n \in \mathbb{N}_0\rangle$ is a Puiseux monoid for all $q \in \mathbb{Q}_{>0}$.

Semirings and Semidomains

Definition: A set S equipped with two binary operations "$+$" and "$\cdot$" is called a semiring if the following conditions hold.

- $(S, +)$ is a monoid with identity element 0.
- $(S, \cdot)$ is a possibly non-cancellative monoid with identity element 1.
- $a \cdot (b + c) = a \cdot b + a \cdot c$ for all $a, b, c \in S$.

Let S be a semiring.

- S is a semidomain if for any $a, b \in S$ with $a \cdot b = 0$, either $a = 0$ or $b = 0$.
- S is a commutative ring if every element in the additive monoid $(S, +)$ is invertible.
- S is an integral domain if S is a commutative ring and a semidomain.

Examples:

1. $\mathbb{N}_0$, $\mathbb{Q}_{\geq 0}$, and $\mathbb{R}_{\geq 0}$ are semidomains under the standard addition and multiplication.
2. The set $\mathbb{N}_0[x]$ of all polynomials with coefficients in $\mathbb{N}_0$ is a semidomain.
3. $\mathbb{Z}$ is an integral domain under the standard addition and multiplication.

Units

Definition: Let M be an additive monoid.

- $u \in M$ is called a unit if there exists $v \in M$ such that $u + v = 0$.
- An abelian group is a monoid where every element is a unit.
- M is called reduced if the only unit of M is 0.

Example: All Puiseux monoids are reduced as 0 is the only unit.

Atoms

Definition: Let M be an additive monoid.

- A nonunit a is called an atom if for any $b, c \in M$ with $a = b + c$, either b or c is a unit.
- $\mathscr{A}(M)$ denotes the set of all atoms of M.
- The monoid M is called antimatter if M has no atoms.
- $b \in M$ is atomic if b is a unit of M or b can be written as a sum of atoms. The monoid M is atomic if every element of M is atomic.

Examples:

1. Every abelian group is antimatter as every element is a unit.
2. The Puiseux monoids $\mathbb{Q}_{\geq 0}$ and $\langle \frac{1}{2^k} : k \in \mathbb{N}_0 \rangle$ are antimatter because every element b can be written as $b = \frac{b}{2} + \frac{b}{2}$.
3. The only atom of $\mathbb{N}_0$ under addition is 1.
4. The multiplicative monoid $\mathbb{N}$ (under the standard multiplication) is an atomic reduced monoid whose set of atoms is $\mathbb{P}$.

Minimal Polynomials

Set $\alpha \in \mathbb{A}$.

Definition: The minimal polynomial $m(x)$ of α is defined as the monic polynomial in $\mathbb{Q}[x]$ of least degree among all polynomials having α as a root.

Definition: A non-constant polynomial is irreducible in $\mathbb{Q}[x]$ if it cannot be factored into the product of two non-constant polynomials in $\mathbb{Q}[x]$.

Examples:

1. The minimal polynomial of $\alpha = \sqrt{2}$ is $x^2 - 2$.
2. The minimal polynomial of $\alpha = \sqrt{3} - \sqrt{2}$ is $x^4 - 10x^2 + 1$.

Descartes' Rule of Signs

Theorem (Descartes' Rule of Signs)

The number of variations of sign of a nonzero polynomial $f(x) \in \mathbb{R}[x]$ has the same parity as and is at least the number of positive roots of $f(x)$ (counting multiplicity).

Theorem (Curtiss, 1918)

For each nonzero polynomial $f(x) \in \mathbb{R}[x]$, there exists a nonzero polynomial $\mu(x) \in \mathbb{R}[x]$ such that the number of variations of sign of $\mu(x)f(x)$ equals the number of positive roots of $f(x)$, counting multiplicity.

Example:

1. As the polynomial $p(x) = x^3 - 2x^2 + 4 \in \mathbb{R}[x]$ has two variations of sign, it follows by Descartes' Rule of Signs, that $p(x)$ has either zero or two positive real roots. Indeed, $p(x)$ has no positive roots as $p(r) \geq 2$ for all $r \in \mathbb{R}_{\geq 0}$.
2. Because $p(x)$ has no positive roots and belongs to $\mathbb{Z}[x]$, the second theorem ensures the existence of a polynomial $\mu(x) \in \mathbb{Z}[x]$ such that $\mu(x)p(x) \in \mathbb{N}_0[x]$ has no variations of sign and so has nonnegative coefficients.

Algebraic Cyclic Semidomains

For $\alpha \in \mathbb{C}$, the subset

$$\mathbb{N}_0[\alpha] := \{p(\alpha) \in \mathbb{C} : p(x) \in \mathbb{N}_0[x]\}.$$

of the field of complex numbers is a semiring. Because $\mathbb{N}_0[\alpha]$ is a subsemiring of a domain, $\mathbb{N}_0[\alpha]$ is a semidomain.

Definition: For $\alpha \in \mathbb{C}$,

- $\mathbb{N}_0[\alpha]$ is called the cyclic semidomain of α, and we say
- $\mathbb{N}_0[\alpha]$ is an algebraic cyclic semidomain (resp., rational cyclic semidomain) provided that α is algebraic (resp., rational).
- M_α denotes the additive monoid $\mathbb{N}_0[\alpha]$ for all $\alpha \in \mathbb{C}$.

Examples:

1. $M_{3/2} = \left\langle \left(\frac{3}{2}\right)^k : k \in \mathbb{N}_0 \right\rangle$ when $\alpha = \frac{3}{2}$.
2. $M_{1/2} = \left\langle \left(\frac{1}{2}\right)^k : k \in \mathbb{N}_0 \right\rangle = \left\{ \frac{n}{2^k} : n, k \in \mathbb{N}_0 \right\}$ when $\alpha = \frac{1}{2}$.
3. $M_{\sqrt{2}} = \left\{ a + b\sqrt{2} : a, b \in \mathbb{N}_0 \right\}$ when $\alpha = \sqrt{2}$.

More About Complex Cyclic Semidomains

Proposition (Correa-Morris-Gotti, 2022)

For transcendental τ, $\mathbb{N}_0[\tau] \cong \mathbb{N}_0[x]$, and $M_\tau \cong \bigoplus_{n\in\mathbb{N}} \mathbb{N}_0$ (the free commutative monoid).

The structure of the free commutative monoid is well studied.

Lemma (Correa-Morris-Gotti, 2022)

If α has a positive conjugate, then $\mathbb{N}_0[\alpha]$ is reduced. Otherwise, $\mathbb{N}_0[\alpha] \cong \mathbb{Z}[\alpha]$.

If $\mathbb{N}_0[\alpha] \cong \mathbb{Z}[\alpha]$, then every element in M_α is a unit, so M_α is trivial from a factorization theoretic perspective.

Hence, we will only study the case where $\alpha \in \mathbb{A} \cap \mathbb{R}_{>0}$.

More about Cyclic Semidomains, cont.

Theorem (Correa-Morris-Gotti, 2022)

Let α be an algebraic number with degree d, minimal polynomial $m_\alpha(x)$, and minimal pair $(p(x), q(x))$. Then the following conditions are equivalent:

1. *M_α is a UFM.*
2. $\deg m_\alpha(x) = |\mathscr{A}(M_\alpha)|$.
3. *$p(x) = x^d$ for some $d \in \mathbb{N}$.*

Moreover, if M_α is a UFM, then it is finitely generated.

Example: For all α such that α is an irreducible n-th root, M_α is a UFM with n atoms.

Atomicity of Cyclic Semidomains

Theorem (Correa-Morris-Gotti, 2022)

The monoid M_α is atomic if and only if $1 \in \mathscr{A}(M)$, and antimatter otherwise. If M_α is atomic, then the atoms are $\{\alpha^i : i \in [0, n) \cap \mathbb{N}_0\}$ for some $n \in \mathbb{N}_0 \cup \{\infty\}$.

Examples:

1. The monoid M_α with $\alpha = \sqrt{2}$ is atomic, with $\mathscr{A}(M_\alpha) = \{1, \sqrt{2}\}$.
2. The monoid M_α with $\alpha = \frac{1}{2}$ is antimatter as $1 \notin \mathscr{A}(M_\alpha)$ $(\frac{1}{2} + \frac{1}{2} = 1)$.
3. The monoid M_α with $\alpha = \sqrt{3/2}$ is atomic with infinitely many atoms: $\mathscr{A}(M_\alpha) = \{\alpha^n : n \in \mathbb{N}_0\}$.

Minimal Polynomials and Atomicity of Cyclic Semidomains

Theorem (D-D-G-L-P-V-Z, 2025)

Let α be a positive algebraic number with minimal polynomial $m(x) \in \mathbb{Q}[x]$. Then the following statements hold.

1. *M_α is finitely generated if and only if $m(x)$ is in $\mathbb{Z}[x]$, the only positive root of $m(x)$ is α, and $|\alpha| \geq |\rho|$ for every root ρ of $m(x)$.*
2. *M_α is antimatter if and only if the only positive root of $m(x)$ is α, the constant term of $m(x)$ after clearing denominators is -1, and $|\rho| \geq |\alpha|$ for every root ρ of $m(x)$.*
3. *In all other cases, M_α is atomic with infinitely many atoms.*

Examples:

1. If $\alpha = \frac{1}{2}$, then $\frac{1}{2}(2x-1)$ is its minimal polynomial and so the monoid M_α is antimatter.
2. If $\alpha = \sqrt{2}$, then $x^2 - 2$ is its minimal polynomial and so the monoid M_α is atomic.
3. If $\alpha = \sqrt{3/2}$, then $\frac{1}{2}(2x^2-3)$ is its minimal polynomial and so the monoid M_α is atomic with infinitely many atoms.

A Canonical Sum Decomposition in M_q

Theorem: Consider the monoid M_q for some $q \in \mathbb{Q}_{>0} \setminus \mathbb{N}$. Then every element $r \in M_q$ can be written uniquely inside M_q as follows:

$$r = c_0(r) + \sum_{n \in \mathbb{N}} c_n(r) q^n. \tag{1}$$

where $(c_n(r))_{n \geq 0}$ is a sequence with nonnegative integer terms, almost all of them being 0, such that $0 \leq c_n(r) < d$ for every $n \in \mathbb{N}$.

Definition: We call the right-hand side of (1) the canonical sum decomposition (CSD) of r inside the monoid M_q.

Examples:

1. In $M_{1/2}$, the CSD of $\frac{11}{4}$ is

$$\frac{11}{4} = 2 \cdot \left(\frac{1}{2}\right)^0 + 1 \cdot \left(\frac{1}{2}\right)^1 + 1 \cdot \left(\frac{1}{2}\right)^2.$$

   The sum decomposition $11 \cdot \left(\frac{1}{2}\right)^2$ is not a CSD as it has too many copies of the summand $\left(\frac{1}{2}\right)^2$.

2. In $M_{3/10}$, one can check that the CSD of $\frac{2023}{1000}$ is

$$\frac{2023}{1000} = 1 \cdot \left(\frac{3}{10}\right)^0 + 2 \cdot \left(\frac{3}{10}\right)^1 + 2 \cdot \left(\frac{3}{10}\right)^2 + 9 \cdot \left(\frac{3}{10}\right)^3.$$

Algorithms for Canonical Sum Decomposition

Theorem (D-D-G-L-P-V-Z, 2025)

For each $q \in \mathbb{Q}_{>0}$, there is an algorithm to find the CSD of an element in M_q in polynomial time.

CSD Algorithm[given $a, b, n, m \in \mathbb{N}$] return CSD of $\frac{a}{b}$ in $M_{n/m}$.

1. Write $\frac{a}{b}$ as $\frac{c}{m^k}$ for minimal $k \in \mathbb{N}_0$.
2. Since c must be in the numerical monoid $\langle m^k, m^{k-1}n, \ldots, n^k \rangle$:

$$\frac{c}{m^k} = \sum_{i=0}^{k} d_i q^i \quad \Longrightarrow \quad c = \sum_{i=0}^{k} d_i m^{k-i} n^i.$$

3. After reducing modulo m, we find that $d_k = c \cdot n^{-k} \pmod{m}$.
4. We subtract $d_k q^k$ from $\frac{c}{m^k}$ and repeat.
5. If any step is not possible, then $\frac{a}{b}$ is not in $M_{n/m}$.

Running the CSD Algorithm

Let us find the CSD of the element $\frac{25}{9}$ in $M_{2/3}$.

1. First, we know that $\frac{25}{9} = c_0 + c_1 \cdot \frac{2}{3} + c_2 \cdot \frac{4}{9}$ for some $c_0, c_1, c_2 \in \mathbb{N}_0$.
2. Clearing denominators, $25 = c_0 \cdot 9 + c_1 \cdot 6 + c_2 \cdot 4$.
3. We find that $c_2 = 25 \cdot 4^{-1} \pmod 3 = 1$.
4. Now we wish to find the CSD of $\frac{25}{9} - 1 \cdot \frac{4}{9} = \frac{7}{3}$.
5. Again, $\frac{7}{3} = c_0 + c_1 \cdot \frac{2}{3}$, so $7 = 3c_0 + 2c_1$.
6. Thus, taking mod 3, we get that $c_1 = 7 \cdot 2^{-1} \pmod 3 = 2$.
7. Subtracting, now we wish to find the CSD of $\frac{7}{3} - 2 \cdot \frac{2}{3} = 1$. The CSD of 1 is 1.
8. After adding them together, we obtain that $1 + 2 \cdot \frac{2}{3} + 1 \cdot \frac{4}{9}$ is the CSD of the element $\frac{25}{9}$ inside $M_{2/3}$

Strong Atoms

Definition: Let M be a monoid.

- An atom $a \in \mathscr{A}(M)$ is called a strong atom if for all $n \in \mathbb{N}_0$, the element na can be written as a sum of atoms in a unique way, namely, as n copies of a.

- $\mathscr{S}(M)$ denotes the set of strong atoms of M.

Definition: Define $f : \mathbb{A} \to (\mathbb{N}_0 \cup \{\infty\})^2$ as $f(\alpha) = (|\mathscr{S}(M_\alpha)|, |\mathscr{A}(M_\alpha)|)$.

A pair $(m, n) \in (\mathbb{N}_0 \cup \{\infty\})^2$ is called realizable if there exists $\alpha \in \mathbb{A}$ such that $f(\alpha) = (m, n)$.

Examples:

1. For any rational $q = a/b$ with $\gcd(a, b) = 1$ and $a, b \geq 2$, the atoms of monoid M_q are $\mathscr{A}(M_q) = \{q^n : n \in \mathbb{N}_0\}$, but M_α has no strong atoms as $\underbrace{q + q + \cdots}_{b \text{ times}} = \underbrace{1 + 1 + \cdots}_{a \text{ times}}$

2. Let α be the root of the polynomial $x^2 - 3x + 1$. The monoid M_α has only one strong atom, which is 1, and α is not a strong atom as $3 \cdot \alpha = 1 + \alpha^2$. Thus, $f(\alpha) = (1, \infty)$.

Rational and n-th Root Cyclic Semidomains

Theorem (Correa-Morris-Gotti, 2022)

For $q \in \mathbb{Q}_{>0}$:

1. *If $q \in \mathbb{N}_0$, then $M_q \cong \mathbb{N}_0$.*
2. *If $q = \frac{1}{b}$ for $b \geq 2$, then M_q is antimatter.*
3. *Otherwise M_q is atomic with $\mathscr{A}(M_q) = \{q^n : n \in M_q\}$.*

Theorem (D-D-G-L-P-V-Z, 2025)

For $q \in \mathbb{Q}_{>0} \setminus \mathbb{N}^{\pm 1}$, let α be a positive irreducible n-th root q. Then M_α is atomic with $\mathscr{A}(M_\alpha) = \{\alpha^n : n \in \mathbb{N}_0\}$ and $\mathscr{S}(M_\alpha) = \emptyset$.

Example: The monoid M_α with $\alpha = \sqrt[3]{5/3}$ is atomic with infinitely many atoms.

Realizable Pairs for Degree-2 Algebraic Numbers

Definition: Define $\mathbb{A}_2$ to be the set of algebraic numbers of degree 2.

Theorem (D-D-G-L-P-V-Z, 2025)

Let α be an algebraic number of degree 2, and let $m(x)$ be the only primitive polynomial that is a scalar multiple of the minimal polynomial of α. Then the following statements hold.

1. *If $m(x) = ax^2 + bx - c$ for some $a, b, c \in \mathbb{N}$, then $f(\alpha) \in \{(0,0), (0,\infty)\}$.*
2. *If $m(x) = ax^2 - bx + c$ for some $a, b, c \in \mathbb{N}$, then $f(\alpha) = (1,\infty)$.*
3. *If $m(x) = ax^2 - bx - c$ for some $a, b, c \in \mathbb{N}$, then $f(\alpha) \in \{(2,2), (2,\infty)\}$.*

Realizable Pairs for Degree-2 Algebraic Numbers, cont.

Examples:

1. If $m(x) = ax^2 + bx - c$ for some $a, b, c \in \mathbb{N}$, then $f(\alpha) \in \{(0,0), (0,\infty)\}$.
   - If α is a root of $2x^2 + 3x - 1$, then $f(\alpha) = (0,0)$ as $1 = 2\alpha^2 + 3\alpha$.
   - If α is a root of $2x^2 + 3x - 2$, then $f(\alpha) = (0,\infty)$ as $2 = 2\alpha^2 + 3\alpha$.
2. If $m(x) = ax^2 - bx + c$ for some $a, b, c \in \mathbb{N}$, then $f(\alpha) = (1,\infty)$.
   - If α is a root of $x^2 - 3x + 1$, then $f(\alpha) = (1,\infty)$ as $3\alpha = \alpha^2 + 1$.
3. If $m(x) = ax^2 - bx - c$ for some $a, b, c \in \mathbb{N}$, then $f(\alpha) \in \{(2,2), (2,\infty)\}$.
   - If α is a root of $x^2 - 3x - 1$, then $f(\alpha) = (2,2)$ as $\alpha^2 = 3\alpha + 1$.
   - If α is a root of $2x^2 - 3x - 1$, then $f(\alpha) = (2,\infty)$ as $2\alpha^2 = 3\alpha + 1$.

Examples of Realizable Pairs

1. Let $\alpha = \sqrt[n]{2}$.

   Since $\alpha^n = 2$, and $1, \alpha, \alpha^2, \ldots, \alpha^{n-1}$ are linearly independent, $\mathscr{A}(M_\alpha) = \mathscr{S}(M_\alpha) = \{\alpha^n : n \in [0, n-1]\}$. Thus, $f(\alpha) = (n, n)$.

2. Let $\alpha = \sqrt{3/2}$.

   Since the minimal polynomial of α is $2x^2 - 3$, M_α has infinitely many atoms. However, since $3 \cdot 1 = 2 \cdot \alpha^2$, 1 is not a strong atom, and so M_α has no strong atoms. Thus, $f(\alpha) = (0, \infty)$.

3. Let α be the root of $x^3 - 8x^2 + 4x - 2$.

   Observe that $(x^2 + 2x + 1)m(x) = x^5 - 6x^4 - 11x^3 - 2x^2 - 2$. Thus, $\alpha^5 = 6\alpha^4 + 11\alpha^3 + 2\alpha^2 + 2$, so there are at most 5 atoms.

   Also, $(2x + 1)m(x) = 2x^4 - 15x^3 - 2$, so $2\alpha^4 = 15\alpha^3 + 2$, and there are at most 4 strong atoms.

   It can be shown that there are exactly 5 atoms and 4 strong atoms, so $f(\alpha) = (4, 5)$.

Constructing Realizable Pairs from Existing Pairs

Theorem (D-D-G-L-P-V-Z, 2025)

Let α be an algebraic number with minimal polynomial $m(x)$ such that $m(x^k)$ is irreducible in $\mathbb{Q}[x]$ for some $k \in \mathbb{N}_{\geq 2}$. If β is a root of $m(x^k)$, then $f(\beta) = kf(\alpha)$.

Theorem (D-D-G-L-P-V-Z, 2025)

If $f(\alpha) = (m, n)$, then the pair (cm, cn) is realizable for every $c \in \mathbb{N}_0$ such that $m_\alpha(x^c)$ is irreducible.

Example: As we've seen, if α is the root of $x^3 - 8x^2 + 4x - 2$, then $f(\alpha) = (4, 5)$. Then, for all $k \in \mathbb{N}$, $\sqrt[k]{\alpha}$ is the root of $x^{3k} - 8x^{2k} + 4x^k - 2$, and $f(\sqrt[k]{\alpha}) = (4k, 5k)$.

Constructing Realizable Pairs

Theorem (D-D-G-L-P-V-Z, 2025)

The pair $(4k + c, 5k + c)$ is realizable for all $(k, c) \in \mathbb{N} \times \mathbb{N}_0$.

For $c = 0$, $f(\alpha) = (4k, 5k)$ where α is the root of $m(x) = x^{3k} - 8x^{2k} + 4x^k - 2$.

For $c > 0$, $f(\alpha) = (4k + c, 5k + c)$ where α is the root of $m(x) = x^{3k+c} - 8x^{2k+c} + 4x^{k+c} - 2x^c - 2$.

This theorem implies the realizability of all (m, n) for all (m, n) with $\frac{m}{n} \geq \frac{4}{5}$.

More Realizable Pairs

Proposition (D-D-G-L-P-V-Z, 2025)

There exists infinitely many $\alpha \in \mathbb{A}$ such that $f(\alpha) = (0, \infty)$.

Proposition (D-D-G-L-P-V-Z, 2025)

For any $\alpha \in \mathbb{A}$ that has 3 positive conjugates (including α), we have $f(\alpha) = (\infty, \infty)$.

Unrealizable Pairs

Lemma (D-D-G-L-P-V-Z, 2025)

If M_α is finitely generated, then $|\mathscr{A}(M_\alpha)| \geq |\mathscr{S}(M_\alpha)| \geq \deg m(x)$, where $m(x)$ is the minimal polynomial of α.

Theorem (D-D-G-L-P-V-Z, 2025)

The pair $(n, n+1)$ is realizable if and only if $n \geq 4$.

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End of Presentation

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